1(a) Show that

$$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4} \qquad (*)$$

Pf: ① when $n=1$

$$1^3=\frac{1^2(1+1)^2}{4}$$

② Suppose when $n=k$ we have $1^3+2^3+\cdots+k^3=\frac{k^2(k+1)^2}{4}$

③ we want prove when $n=k+1$ the equation $(*)$ is also right.

$\Leftrightarrow$ we want $1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{(k+1)^2(k+1+1)^2}{4}$

$$LHS=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2(k^2+4(k+1))}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

then we have $LHS=RHS$

by Mathematical Induction, we have proven this equation.

(b) Show that

$$C_0^{a+0}+C_1^{a+1}+\cdots+C_n^{a+n}=C_n^{a+n+1} \qquad (*)$$

Pf: ① when $n=0$ $\qquad C_0^{a+0}=1=C_0^{a+0+1}$ by definition

② Suppose $n=k$ (*) is right

that is $C_0^{a+0} + C_1^{a+1} + \cdots + C_k^{a+k} = C_k^{a+k+1}$

③ we want to prove when $n=k+1$. (*) is also right.

$\Rightarrow$ we want $C_0^{a+0} + C_1^{a+1} + \cdots + C_k^{a+k} + C_{k+1}^{a+k+1} = C_{k+1}^{a+k+1+1}$

by ② we have $LHS = C_k^{a+k+1} + C_{k+1}^{a+k+1}$

$$\overset{\text{by definition}}{=} \frac{(a+k+1)\cdot(a+k+1-1)\cdots(a+k+1-k+1)}{k!}$$

$$+ \frac{(a+k+1)(a+k+1-1)\cdots(a+k+1-(k+1)+1)}{(k+1)!}$$

$$\Rightarrow LHS = \frac{(a+k+1)\cdot(a+k+1-1)\cdots(a+2)}{k!} + \frac{(a+k+1)\cdots(a+1)}{(k+1)!}$$

$$= \frac{(a+k+1)(a+k+1-1)\cdots(a+2)(k+1) + (a+k+1)\cdots(a+1)}{(k+1)!}$$

$$= \frac{[(a+k+1)\cdots(a+2)](k+1+a+1)}{(k+1)!}$$

$$= \frac{(a+k+2)(a+k+1)\cdots(a+2)}{(k+1)!}$$

$$RHS \overset{\text{by definition}}{=} \frac{(a+k+2)(a+k+2-1)\cdots(a+k+2-(k+1)+1)}{(k+1)!}$$

$$RHS = \frac{(a+k+2)(a+k+2-1)\cdots(a+2)}{(k+1)!}$$

So we have $LHS = RHS$

by Mathematical Induction, we have proven this equation

2. $f: (0, \infty) \to (0, \infty)$

$f(x) = \ln(1+x^2)$ Show that $R(f) = (0, \infty)$

Sol: let $u = 1+x^2$ so $u \in (1, \infty)$ since $x \in (0, \infty)$

let $g(u) = \ln u$ then $f(x) = g(u)$

and $R(f) = R(g)$

let $h(y) = e^y : (0, \infty) \to (1, \infty)$

$h(y)$ is an inverse function of $g(u)$ since $\exp(\ln(u)) = u$

So $R(g) = (0, \infty)$

3. Sol: let $u = 1+x^2$ then $u \in (1, \infty)$ let $g(u) = \ln u$

then $f(x) = g(u)$

let $h(y) = e^y : (0, \infty) \to (1, \infty)$

$h(y)$ is an inverse function of $g(u)$ since $\exp(\ln(u)) = u$.

$\Rightarrow g(u)$ is an injective function.

$u = 1+x^2 \Rightarrow x = \sqrt{u-1}$ and the square root of $\sqrt{u-1}$ is always defined $\Rightarrow u = 1+x^2$ is an injective function $\Rightarrow f(x)$ is also injective.

4. Sol ①: $g(x) = x^3 + bx$ then $g'(x) = 3x^2 + b$

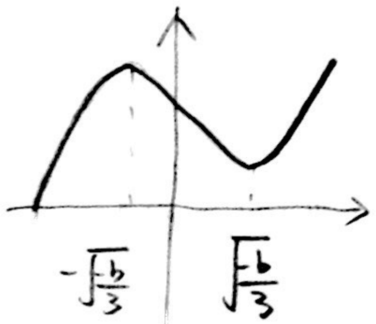
Case I: If $b \geq 0$ $g'(x) \geq 0$ then $g(x)$ is an increasing function and only when $x=0$ we have $g'(x) = 0$

when $x \neq 0$ $g'(x) > 0$

$\Rightarrow g(x)$ is an injective function.

Case II: If $b < 0$ then we have

$(-\infty, -\sqrt{\frac{-b}{3}})$	$[-\sqrt{\frac{-b}{3}}, \sqrt{\frac{-b}{3}}]$	$(\sqrt{\frac{-b}{3}}, \infty)$
$g'(x) > 0$	$g'(x) \leq 0$	$g'(x) > 0$



$g(x)$ is not an injective function

So we have when $b \geq 0$ $g(x)$ is an injective function

Sol ②: we want $g(x) = x^3 + bx$ is an injective function.

Suppose $\exists\, x_1\ x_2$ s.t. $g(x_1) = g(x_2)$

$\Rightarrow x_1^3 + bx_1 = x_2^3 + bx_2$

$\Rightarrow (x_1^3 - x_2^3) + (bx_1 - bx_2)$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + b) = 0$

we want $x_1 - x_2 = 0$ So we need $x_1^2 + x_1x_2 + x_2^2 + b \neq 0$

And we have $x_1^2 + x_1x_2 + x_2^2 + b = x_1^2 + x_1x_2 + \frac{1}{4}x_2^2 + \frac{3}{4}x_2^2 + b$

$= \underbrace{(x_1+\frac{1}{4}x_2)^2}_{\geq 0} + \underbrace{\frac{3}{4}x_2^2}_{\geq 0} + b$

we need $b \geq 0$ then the whole expression ≥ 0

when $(x_1+\frac{1}{4}x_2)^2+\frac{3}{4}x_2^2+b=0 \Rightarrow x_1=x_2=0$

it also have $x_1=x_2$

So $b\geq 0$ can satisfy the condition

For the case $\forall b<0$ can find $x_1 x_2$ s.t. $x_1 \neq x_2$

and $(x_1+\frac{1}{4}x_2)^2+\frac{3}{4}x_2^2+b=0$

So in a word only when $b \in [0, \infty)$ can make $g(x)$ an injective function

5. $x_1 > x_2 > 0$ we want to show $\exp(x_1) > \exp(x_2)$

$\Leftrightarrow \frac{\exp(x_1)}{\exp(x_1)} > 1$ and we have $\exp(x_1-x_2) = \frac{\exp(x_1)}{\exp(x_2)}$

$\Leftrightarrow \exp(x_1-x_2) > 1$

and $\exp(x_1-x_2) = 1 + \frac{x_1-x_2}{1!} + \frac{(x_1-x_2)^2}{2!} + \dots$

> 1

Since $\forall \frac{(x_1-x_2)^n}{n!} > 0$ because $x_1-x_2>0$

So we have prooved $\exp(x_1) > \exp(x_2)$

6. $x_2 < x_1 < 0$

we want to show $\exp(x_1) > \exp(x_2)$

$\Leftrightarrow \frac{\exp(x_1)}{\exp(x_2)} > 1$ since $\exp(x_1 - x_2) = \frac{\exp(x_1)}{\exp(x_2)}$

$\Leftrightarrow \exp(x_1 - x_2) > 1$

and $\exp(x_1 - x_2) = 1 + \frac{x_1 - x_2}{1!} + \frac{(x_1 - x_2)^2}{2!} + \cdots$

$\forall \frac{(x_1 - x_2)^n}{n!} > 0$ since $x_1 - x_2 > 0$

So $\exp(x_1 - x_2) > 1$

$\Rightarrow \exp(x_1) > \exp(x_2)$